Mid-Semester Exam ISI Bangalore March 2019

Introduction to Algebraic Geometry 100 Points

Notes.

(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.

(b) There are a total of 120 points in the paper. You will be awarded a maximum of 100.

(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(c) By default, k denotes an algebraically closed field and $\mathbb{A}^n_k$ is the affine n-space over k while $\mathbb{P}^n_k$ is the projective n-space over k. By default, the polynomial ring of functions on $\mathbb{A}^n_k$ is denoted as $k[x_1, \ldots, x_n]$ while for $n = 1, 2, 3$ we also use the usual notation of x, y, z for the variables.

(d) We will use $\mathcal{V}(-)$ to denote the common zero locus (in suitable affine space) of any collection of polynomials and $\mathcal{I}(-)$ the ideal of functions vanishing on a given subset of affine space.

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1. [15 points] Let Z denote the image of the polynomial map $\mathbb{A}^1_k \to \mathbb{A}^2_k$ given by $t \mapsto (t^3, t^4)$.
   (i) Verify that Z is a closed subset of $\mathbb{A}^2_k$.
   (ii) Verify that the natural induced map $f \colon \mathbb{A}^1_k \to Z$ is a bijection.
   (iii) Prove that f is not an isomorphism.

2. [15 points] Let $p(t), q(t)$ be polynomials of degree m, n respectively. Let C denote the affine plane curve given by $p(x) - q(y) = 0$. Assume that the characteristic of the field k is 0. Find the number of points at ∞ of C.
(Hint: The answer varies according to whether $m = n$ or not.)

3. [15 points] Let X be an affine algebraic set. Prove that a basis for the Zariski topology on X is given by the open sets of the form X_f as f ranges over all polynomial functions on X where X_f denotes the locus $f \neq 0$ in X.

4. [15 points] Find the Zariski closure of the following subsets of affine space.
   (i) $\{(n, n^2) \mid n \in \mathbb{Z}\} \subset \mathbb{A}^2_{\mathbb{C}}$.
   (ii) The set of all invertible matrices in $\mathbb{A}^{n^2}_k$ where $\mathbb{A}^{n^2}_k$ is naturally identified with the set of all $n \times n$ matrices over k.
   (iii) The locus $z = \sin(w)$ in $\mathbb{A}^2_{\mathbb{C}}$.

5. [15 points] Find the irreducible components and the connected components of the algebraic set $\mathcal{V}(x^2(y^2 - x^4)^3, (y-1)^4(y^2 - x^4)^5) \subset \mathbb{A}^2_k$.

6. [15 points] Let R be a noetherian ring. Suppose $I \subset R$ is an ideal such that there is only one prime ideal, say $\mathfrak{p}$, containing I. Prove the following:
   (i) $R_{\mathfrak{p}}/I_{\mathfrak{p}} \xrightarrow{\sim} R/I$.

(ii) There exists an integer $N > 0$ such that $\mathfrak{p}^N \subset I$.

7. [15 points] Find the points of intersection of the affine plane curves $y - x^2 = 0$ and $y^2 - 2x^2 - x^3 = 0$ and calculate the intersection multiplicity at each point of the intersection.

8. [15 points] Let R be a k-algebra and let f be an element of R.
   (i) Prove that if R is finitely generated as a k-algebra, then $R_f = \cap R_{\mathfrak{m}}$ where the intersection ranges over all the maximal ideals $\mathfrak{m}$ containing f.
   (ii) Give an example of a k-algebra R and an element f such that the conclusion of (i) fails.